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Nested T-duality

C. Klimčík

Institute de mathématiques de Luminy,
163, Avenue de Luminy, 13288 Marseille, France

Abstract

We identify the obstructions for T-dualizing the boundary WZW model and make explicit how they depend on the geometry of branes. In particular, the obstructions disappear for certain brane configurations associated to non-regular elements of the Cartan torus. It is shown in this case that the boundary WZW model is "nested" in the twisted boundary WZW model as the dynamical subsystem of the latter.

1 Introduction

Some time ago, Kiritsis and Obers have proposed a T-dualization of the closed string WZW model, based on the existence of an outer automorphism of the group target G . The purpose of this article is to work out a qualitative remark in [1] where it was suggested that there should exist an open string version of the Kiritsis-Obers T-duality [2]. In order words, the ordinary and the twisted boundary WZW models should be isomorphic as dynamical systems. We remind that the former (latter) describes the dynamics of WZW open strings whose end-points stick on ordinary (twisted) conjugacy classes therefore the existence of such T-duality would establish quite a non-trivial relation between branes with different dimensions and geometries. For generic branes, it turns out, however, that the duality can be established only at the price of constraining some zero modes on each side of the dual pair of models. This is the usual state of matters known from the non-Abelian T-duality story for closed strings [3, 4] and it therefore seems that, generically, the open strings do not perform better as their closed counterparts. Nevertheless, there is a subtle difference between the open and closed cases hidden in the term "generic". In fact, there are exceptional branes for which the duality can be established at least in one direction, i.e. there is no need to constrain the zero modes on one side of the T-dual pair of models. This is a new phenomenon which never occurs for closed strings and we refer to it as to the nested or one-way duality, since, in this case, the full-fledged non-constrained boundary model is the physical subsystem of its T-dual. We shall see that this phenomenon occurs for certain branes associated with the non-regular elements of the Cartan torus and it is related to the fact that the centralizers of these elements are sufficiently big.

The plan of the paper is as follows: first we rewrite the known symplectic structure of the (twisted) boundary WZW model in new variables which single out the zero modes causing the obstruction to duality. Then we show that in some cases a small gauge symmetry, coming from the bigger centralizers of non-regular elements, happens to gauge away the obstructional zero modes and it embeds the boundary WZW model into its twisted counterpart.

2 Review of the bulk WZW model

Usually, the symplectic structure of the bulk and also of the boundary WZW model is derived from the second-order least action principle as in [7, 8, 9]. We shall not repeat this derivation here for the sake of conciseness and we shall rather stick on the Hamiltonian formalism from the very beginning. Let LG be the loop group of a Lie group G . The bulk WZW model is a dynamical system whose phase space points are pairs (g, J) , $g \in LG$, $J \in \text{Lie}(LG)$ and the symplectic form of which reads (see [5, 6]):

$$\omega = d(J, dgg^{-1})_{\mathcal{G}} + \frac{1}{2}k(dgg^{-1} \wedge \partial_{\sigma}(dgg^{-1}))_{\mathcal{G}}. \quad (1)$$

Here dgg^{-1} is a $\text{Lie}(G)$ -valued right-invariant Maurer-Cartan form on LG , ∂_{σ} is the derivative with respect to the loop parameter σ and

$$(\chi_1, \chi_2)_{\mathcal{G}} \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma \text{Tr}(\chi_1(\sigma)\chi_2(\sigma)).$$

The dynamics of the bulk WZW model is then given by a (Hamiltonian) vector field v defining the time evolution. Explicitely,

$$v = \int_{-\pi}^{\pi} \partial_{\sigma} J(\sigma) \frac{\delta}{\delta J(\sigma)} + \frac{1}{k} \nabla_{J_L}^L - \frac{1}{k} \nabla_{J_R}^R, \quad (2)$$

where

$$J_L \equiv J, \quad J_R = -g^{-1}Jg + kg^{-1}\partial_{\sigma}g.$$

Remind that, for $\chi \in \text{Lie}(LG)$, the differential operators ∇_{χ}^L , ∇_{ψ}^R acts on functions on the loop group LG as follows

$$\nabla_{\chi}^L f(g) = \frac{d}{dt} f(e^{\chi}g)|_{t=0}, \quad \nabla_{\psi}^R f(g) = \frac{d}{dt} f(ge^{\chi})|_{t=0}.$$

For completeness, the vector field v fulfills the relation $\iota_v \omega \equiv \omega(v, .) = dH$ where the hamiltonian H is given by (cf.[6])

$$H = -\frac{1}{4\pi k} \int_{-\pi}^{\pi} d\sigma \text{Tr}(J_L^2 + J_R^2).$$

Finally, let us note that the observables J_L and J_R generate (via the Poisson bracket) the left and right action of the loop group LG on the phase space of the model.

3 Branes in the symplectic language

Denote SG a "segment" group consisting of smooth maps from an interval $[0, \pi]$ into a (simple, connected, simply connected) compact Lie group G and consider a space SP of pairs (g, J) where $g \in SG$ and $J \in \text{Lie}(SG)$. It makes thus sense to write down a two-form ω_s and a vector field v_b on SP , given, respectively, by the following formulas:

$$\omega_s = d(J, dgg^{-1})_b + \frac{1}{2}k(dgg^{-1} \wedge \partial_\sigma(dgg^{-1}))_b. \quad (3)$$

$$v_b = \int_0^\pi \partial_\sigma J(\sigma) \frac{\delta}{\delta J(\sigma)} + \frac{1}{k} \nabla_{J_L}^L - \frac{1}{k} \nabla_{J_R}^R. \quad (4)$$

Here dgg^{-1} is a $\text{Lie}(SG)$ -valued right-invariant Maurer-Cartan form on SG , J_L and J_R are defined as before and $(., .)_b$ stands for

$$(\chi_1, \chi_2)_b \equiv \frac{1}{\pi} \int_0^\pi d\sigma \text{Tr}(\chi_1(\sigma) \chi_2(\sigma)).$$

The "segment" formulae (3),(4) resemble "loop" formulae (1), (2) but they do not define a dynamical system since the form ω_s is not closed. Indeed, a simple book-keeping of boundary terms gives

$$d\omega_s = -\frac{k}{6\pi} \text{Tr}(dg(\pi)g(\pi)^{-1})^3 + \frac{k}{6\pi} \text{Tr}(dg(0)g(0)^{-1})^3 \quad (5)$$

In order to modify ω_s and render it closed, we have to remind some standard concepts from the Lie group theory:

Let ξ be an involutive automorphism of G , which respects also the Killing-Cartan form $\text{Tr}(., .)$ on $\text{Lie}(G)$. Consider then two fixed elements $m_\pi, m_0 \in G$ and their corresponding ξ -conjugacy classes $C_{m_\pi}^\xi, C_{m_0}^\xi$ in G . Recall that

$$C_m^\xi = \{g \in G; \exists s \in G, g = \xi(s)ms^{-1}\}.$$

To the element $m \in G$ and to the automorphism ξ , we can associate a two-form on G given by

$$\tilde{\alpha}_m^\xi = 2\text{Tr}(\xi(s^{-1}ds)ms^{-1}dsm^{-1}), \quad (6)$$

where we denoted the elements of G by the symbol s . We define also a ξ -centralizer as $\text{Cent}^\xi(m) = \{t \in G; \xi(t)mt^{-1} = m\}$. It is easy to check that the form $\tilde{\alpha}_m^\xi$ is degenerated in the directions of the right action of $\text{Cent}^\xi(m)$ on G hence $\tilde{\alpha}_m^\xi$ is a pull-back (under the map $s \rightarrow \xi(s)ms^{-1}$) of certain form α_m^ξ defined on the ξ -conjugacy class C_m^ξ [10, 11, 8].

Starting from the formula (6), it is easy to find out that

$$d\alpha_m^\xi = \frac{2}{3}\text{Tr}(dgg^{-1})^3, \quad (7)$$

where $g = \xi(s)ms^{-1}$.

The formulae (5) and (7) motivate us to define a dynamical system $(M^\xi, \omega_{m_\pi m_0}^\xi, v_b)$ whose phase space is the submanifold M^ξ of SP such that

$$g(0) \in C_{m_0}^\xi, \quad g(\pi) \in C_{m_\pi}^\xi, \quad (8)$$

$$\xi(J_R(0)) = J_L(0), \quad \xi(J_R(\pi)) = J_L(\pi), \quad (9)$$

the symplectic form of which is

$$\omega_{m_\pi m_0}^\xi = d(J, dgg^{-1})_b + \frac{1}{2}k(dgg^{-1} \wedge \partial_\sigma dgg^{-1})_b + \frac{k}{4\pi}\alpha_{m_\pi}^\xi(g(\pi)) - \frac{k}{4\pi}\alpha_{m_0}^\xi(g(0))$$

and the evolution vector field of which is

$$v_b = \int_0^\pi \partial_\sigma J(\sigma) \frac{\delta}{\delta J(\sigma)} + \frac{1}{k} \nabla_{J_L}^L - \frac{1}{k} \nabla_{J_R}^R.$$

It is clear that the boundary conditions (8) (restricting the end-points of the string on the ξ -conjugacy classes) make the form $\omega_{m_\pi m_0}^\xi$ closed. The glueing conditions (9) are dictated by the requirement: that the evolution vector field v_b must respect the boundary conditions (8). It turns out that v_b is a hamiltonian vector field, i.e. $\iota_{v_b}\omega_{m_\pi m_0}^\xi = dH_b$ with

$$H_b = -\frac{1}{2\pi k} \int_0^\pi \text{Tr}(J_L^2 + J_R^2).$$

This can be checked by using the following identity (cf. [12]):

$$\iota_{v_b}\alpha_{m_0}^\xi = -\frac{2}{k}\text{Tr}(J_L(0), dg(0)g(0)^{-1} + \xi(g(0)^{-1}dg(0))).$$

4 Loop group parametrization

Now we establish that the dynamical system $(M^\xi, \omega_{m_\pi m_0}^\xi, v_b)$ introduced above is nothing but the boundary WZW model described in [8]. For this, we first replace the coordinates $J(\sigma), g(\sigma)$ on M^ξ by a pair of group valued variables $g_R(\sigma), g_L(\sigma)$ as follows

$$J = -k\xi(g_R^{-1}\partial_\sigma g_R), \quad g = \xi(g_R^{-1})g_L.$$

In the new parametrization, the symplectic form $\omega_{m_\pi m_0}^\xi$ becomes

$$\begin{aligned} \omega_{m_\pi m_0}^\xi &= \frac{1}{2}k(dg_L g_L^{-1} \wedge \partial_\sigma(dg_L g_L^{-1}))_b - \frac{1}{2}k(dg_R g_R^{-1} \wedge \partial_\sigma(dg_R g_R^{-1}))_b \\ &\quad - \frac{k}{2\pi} \text{Tr}(\xi(dg_R g_R^{-1}) \wedge dg_L g_L^{-1}) \Big|_0^\pi + \frac{k}{4\pi} \alpha_\mu^\xi(\xi(g_R^{-1})g_L) \Big|_0^\pi \end{aligned}$$

Note that the form $\omega_{m_\pi m_0}^\xi$ is now degenerated along the vector fields corresponding to the (gauge) transformations $(g_R(\sigma), g_L(\sigma)) \rightarrow (\xi(A)g_R(\sigma), Ag_L(\sigma))$ with $A \in G$. This degeneracy expresses the ambiguity of the replacing (J, g) by (g_R, g_L) . The boundary and the glueing conditions now read

$$g_L(\pi)g_R^{-1}(\pi) \in C_{m_\pi}^\xi, \quad g_L(0)g_R^{-1}(0) \in C_{m_0}^\xi, \quad (10)$$

$$(g_R^{-1}\partial_\sigma g_R)(\pi) = -(g_L^{-1}\partial_\sigma g_L)(\pi), \quad (g_R^{-1}\partial_\sigma g_R)(0) = -(g_L^{-1}\partial_\sigma g_L)(0). \quad (11)$$

Set

$$H_\pi \equiv g_L(\pi)g_R(\pi)^{-1}, \quad H_0 \equiv g_L(0)g_R(0)^{-1}, \quad G_0 e^{2\pi\nu} G_0^{-1} = H_\pi H_0^{-1}$$

where ν is in the Weyl alcove A_+ . We now introduce a G -valued field $h(\sigma)$, $\sigma \in [-\pi, \pi]$ as follows

$$h(\sigma) = g_L(\sigma)^{-1}G_0e^{\nu\sigma}, \quad \text{for } \sigma \in [0, \pi],$$

$$h(\sigma) = g_R(-\sigma)^{-1}H_0^{-1}G_0e^{\nu\sigma}, \quad \text{for } \sigma \in [-\pi, 0],$$

In the new parametrization, the boundary conditions (10) say that $h(\sigma)$ is a continuous loop, i.e. $h(-\pi) = h(\pi)$, and the glueing conditions (11) insure

that $h(\sigma)$ is moreover smooth (in particular in 0 and in π). The form $\omega_{m_\pi m_0}^\xi$ becomes

$$\begin{aligned}\omega_{m_\pi m_0}^\xi = & k(h^{-1}dh \wedge \partial_\sigma(h^{-1}dh))_G + 2kd(\nu, h^{-1}dh)_G + \\ & + 2k\text{Tr}(G_0^{-1}dG_0 \wedge d\nu) + \frac{k}{2\pi}\text{Tr}(H_\pi^{-1}dH_\pi \wedge H_0^{-1}dH_0) + \\ & + \frac{k}{4\pi}\alpha_{m_\pi}^\xi(H_\pi) - \frac{k}{4\pi}\alpha_{m_0}^\xi(H_0) - \frac{k}{2\pi}\text{Tr}(G_0^{-1}dG_0 e^{2\pi\nu} G_0^{-1}dG_0 e^{-2\pi\nu}).\end{aligned}\quad (12)$$

The degeneracy directions $(g_R(\sigma), g_L(\sigma)) \rightarrow (\xi(A)g_R(\sigma), Ag_L(\sigma))$ of $\omega_{m_\pi m_0}^\xi$ become now $(h(\sigma), \nu, H_\pi, H_0, G_0) \rightarrow (h(\sigma), \nu, AH_\pi\xi(A)^{-1}, AH_0\xi(A)^{-1}, AG_0)$. The form $\omega_{m_\pi m_0}^\xi$ given by (12) coincides with the symplectic form of the boundary WZW model reported in the paper [8]. The careful reader will notice, however, that the role of our boundary $\sigma = 0$ is played by $\sigma = \pi$ in [8].

5 Obstruction to duality

The loop group representation (12) shows that models with different ξ may differ just in the zero mode sector. However, it is not easy to compare them directly since the ranges of the zero modes (H_0, H_π, G_0, ν) also differ for different ξ . We shall therefore make once again a suitable change of parametrization of the phase space of the ξ -boundary model. We first use a formula

$$\begin{aligned}\alpha_\mu^\xi(\xi(g)^{-1}hg) = & \alpha_\mu^\xi(h) + 2\text{Tr}(\xi(dgg^{-1}) \wedge hdgg^{-1}h^{-1}) - 2\text{Tr}(h^{-1}dh \wedge dgg^{-1} + dh h^{-1} \wedge \xi(dgg^{-1}))\\ \text{and then rewrite the form } \omega_{m_\pi m_0}^\xi \text{ as follows:}\end{aligned}$$

$$\begin{aligned}\omega_{m_\pi m_0}^\xi = & \frac{1}{2}k(dg_L g_L^{-1} \wedge \partial_\sigma(dg_L g_L^{-1}))_b - \frac{1}{2}k(dg_R g_R^{-1} \wedge \partial_\sigma(dg_R g_R^{-1}))_b \\ & + \frac{k}{2\pi}\text{Tr}\left(g_\pi^{-1}dg_\pi \wedge (\xi(dg_L(\pi)g_L(\pi)^{-1}) - dg_R(\pi)g_R(\pi)^{-1})\right) \\ & - \frac{k}{2\pi}\text{Tr}\left(g_0^{-1}dg_0 \wedge (\xi(dg_L(0)g_L(0)^{-1}) - dg_R(0)g_R(0)^{-1})\right).\end{aligned}\quad (13)$$

Here the variables g_0 , g_π are defined by

$$g_L(\pi)g_R(\pi)^{-1} = \xi(g_\pi)^{-1}m_\pi g_\pi, \quad g_L(0)g_R(0)^{-1} = \xi(g_0)^{-1}m_0 g_0. \quad (14)$$

Set

$$\begin{aligned}\tilde{g}_R(\sigma) &= p^{-1}e^{\tau\sigma}pg_0g_R(\sigma) \\ \tilde{g}_L(\sigma) &= \xi(p^{-1}e^{\tau\sigma}pg_0)g_L(\sigma)\end{aligned}$$

where $p \in G$, $\tau \in A_+$ are defined as

$$p^{-1}e^{\tau\pi}p = g_\pi g_0^{-1}. \quad (15)$$

The symplectic form $\omega_{m_\pi m_0}^\xi(x, y)$ then becomes

$$\begin{aligned}\omega_{m_\pi m_0} = \frac{1}{2}k(d\tilde{g}_L\tilde{g}_L^{-1} \wedge \partial_\sigma(d\tilde{g}_L\tilde{g}_L^{-1}))_b - \frac{1}{2}k(d\tilde{g}_R\tilde{g}_R^{-1} \wedge \partial_\sigma(d\tilde{g}_R\tilde{g}_R^{-1}))_b + \\ + kd(\xi(\chi), d\tilde{g}_L\tilde{g}_L^{-1})_b - kd(\chi, d\tilde{g}_R\tilde{g}_R^{-1})_b,\end{aligned}$$

where

$$\chi = p^{-1}\tau p. \quad (16)$$

The Hamiltonian now reads

$$H_b^\xi = -\frac{k}{2\pi} \int_0^\pi d\sigma \text{Tr} \left((\partial_\sigma \tilde{g}_L \tilde{g}_L^{-1} - \xi(\chi))^2 + (\partial_\sigma \tilde{g}_R \tilde{g}_R^{-1} - \chi)^2 \right).$$

The boundary and glueing conditions (10, 11) become, respectively,

$$\begin{aligned}\tilde{g}_L(0) &= m_0 \tilde{g}_R(0), \quad \tilde{g}_L(\pi) = m_\pi \tilde{g}_R(\pi), \\ \left(\tilde{g}_L^{-1} \partial_\sigma \tilde{g}_L - \tilde{g}_L^{-1} \xi(\chi) \tilde{g}_L \right)_{\sigma=0,\pi} &= - \left(\tilde{g}_R^{-1} \partial_\sigma \tilde{g}_R - \tilde{g}_R^{-1} \chi \tilde{g}_R \right)_{\sigma=0,\pi}.\end{aligned}$$

In the last parametrization, we observe that the range of the new variables $(\tilde{g}_L, \tilde{g}_R, \chi)$ is the same for all automorphisms ξ , in particular $e^{\pi\chi}$ sweeps the whole group G . This means that the observable χ is the obstruction for T-duality since, in general, we have $\xi_1(\chi) \neq \xi_2(\chi)$ for $\xi_1 \neq \xi_2$. Indeed, the T-duality means the dynamical equivalence of two models, therefore it cannot be established unless we impose the constraint $\xi_1(\chi) = \xi_2(\chi)$ for both models. It seems that we cannot do better and the T-duality takes place only between the pair of the *constrained* models. Generically, this conclusion is correct but there are particular brane geometries for which the constraint $\xi_1(\chi) = \xi_2(\chi)$ is satisfied dynamically and need not be imposed by hand. We shall explain in the next section how this happens and we shall interpret the phenomenon as the nested duality.

6 Non-regular branes

It is important to notice that, so far, we have been somewhat abusing the terminology by calling "symplectic" also those 2-forms which might have been only presymplectic. Here by adjective "presymplectic" we mean a 2-form which is closed but need not be non-degenerate. The possible degeneracy of a closed 2-form ω does not mean that we do not deal with a honest dynamical system but it rather indicates that we study the system with a gauge symmetry. Infinitesimally, the gauge symmetry is generated by vector fields w on the phase space that annihilate the symplectic form, i.e. such that $\iota_w \omega = 0$. It is the closedness of the form ω then insures that the Lie bracket $\{w_1, w_2\}$ of two annihilating vector fields is again an annihilating vector field.

In our context, the degeneracy gauge group $K_{\mu_\pi \mu_0}^\xi$ depends on the brane geometry and also on the model attributed to ξ and it is given as the direct product of the twisted centralizers, i.e. $K_{m_\pi m_0}^\xi = \text{Cent}_{m_\pi}^\xi \times \text{Cent}_{m_0}^\xi$. This fact can be easily seen from (14) since it expresses the ambiguity of the definition of the variables g_π, g_0 . Indeed, the pair $(t_\pi^\xi g_\pi, t_0^\xi g_0)$ is equivalent to (g_0, g_π) for $t_{\pi,0}^\xi \in \text{Cent}_{m_\pi,0}^\xi$ and the reader may check by direct calculation the corresponding degeneracy of the symplectic form (13). Now the variable χ transforms nontrivially under the gauge transformations:

$$e^{\pi\chi} \rightarrow t_\pi^\xi e^{\pi\chi} (t_0^\xi)^{-1}, \quad (17)$$

as it can be seen from Eqs. (15) and (16).

We ask the following question: If we work with the ξ_1 -model, can we achieve the equality $\xi_1(\chi) = \xi_2(\chi)$ by a K^{ξ_1} -gauge transformation? If yes, then we see immediately that the ξ_1 -boundary WZW model is the dynamical subsystem of the ξ_2 -boundary WZW model selected by the constraint $\xi_2(\chi) = \xi_1(\chi)$ imposed on the *latter*. Similarly, if we work with the ξ_2 -model and we achieve the equality $\xi_1(\chi) = \xi_2(\chi)$ by K^{ξ_2} -gauge transformation we conclude that the ξ_2 -model is the subsystem of the ξ_1 -model. Of course, the ideal situation would be, if for some choice m_π, m_0 the equality $\xi_1(\chi) = \xi_2(\chi)$ could be reached by both K^{ξ_1} and K^{ξ_2} gauge transformations. This would mean that the ξ_1 -model and ξ_2 model would be strictly dynamically equivalent or, in other words, T-dual to each other. Unfortunately, we did not find such a configuration m_π, m_0 and we even conjecture that it does not exist. Nevertheless, the situation when one model is the submodel of the other does happen for the so called non-regular branes. In what follows, we shall

illustrate the ideas and concepts of this section on the case of the group $G = SU(3)$. The first automorphism $\xi_1 = id$ will be just trivial identity map and the second $\xi_2 = cc$ will be given by the standard complex conjugation.

We start with the twisted case (i.e. $\xi_2 = cc$) and we wish to find a gauge transformation which would bring any value of the variable χ onto the slice $\chi^{cc} = \chi$. This means that the gauge group $K_{m_\pi m_0}^{cc}$ has to be at least 5-dimensional. Indeed, the subgroup of cc -invariant elements (i.e. real matrices) in $SU(3)$ is three dimensional while the dimension of $SU(3)$ itself is eight. The possible twisted centralizers $\text{Cent}^{cc}(m)$ have been classified by Stanciu in [13]. They are mostly one-dimensional but in two cases they have dimension three. In particular, it is the cc -conjugacy class of the unit element for which $\text{Cent}^{cc}(e) = SO(3)$ and the cc -conjugacy class of the element $f \in SU(3)$

$$f = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for which $\text{Cent}^{cc}(f) = SU(2)$. Thus the counting of dimensions says that it is necessary for both string end-points to live either on $C^{cc}(e)$ or on $C^{cc}(f)$ if we want to get to the slice $\chi^{cc} = \chi$ by the gauge transformation. Unfortunately, this condition is not sufficient. Indeed, the centralizer of the unit element does not take χ away from the slice (since it is real) therefore it cannot bring χ on the slice either. It remains only the case when both string end-points live on $C^{cc}(f)$. In this case $\text{Cent}^{cc}(f)$ is embedded in $SU(3)$ in the following way:

$$\begin{pmatrix} a & b & 0 \\ -\bar{b} & \bar{a} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a\bar{a} + b\bar{b} = 1. \quad (18)$$

It is clear that a one-parameter group of real matrices

$$\begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is contained in $\text{Cent}^{cc}(f)$ and, at the same time, it cannot help to bring χ on the slice. Thus we loose one parameter at each end-point and the four remaining parameters from two copies of $\text{Cent}^{cc}(f)$ cannot bring the 8-dimensional object χ onto the 3-dimensional slice $\chi^{cc} = \chi$. We conclude that

the twisted cc -boundary WZW model is never a subsystem of the ordinary id -boundary WZW model, whatever is the choice of the cc -conjugacy classes.

We now show, on the contrary, that the ordinary id -boundary WZW model can be the subsystem of the cc -boundary WZW model or, in other words, the nested duality can take place. Indeed, the slice $\chi^{cc} = \chi$ can be achieved by the gauge transformation from $K_{m_\pi m_0}^{id}$ for several conjugacy classes $C_{m_\pi}^{id}$ and $C_{m_0}^{id}$. In particular, if we consider a point-like brane C_c^{id} where the element c is from the $SU(3)$ center, we have $\text{Cent}^{id}(c) = SU(3)$ and the formula (xy) says that the gauge transformation can bring any χ on the slice $\chi^{cc} = \chi$. There is another example, which is perhaps more interesting. Consider a fixed real number ϕ and the following element of $SU(3)$.

$$m_\phi = \begin{pmatrix} e^{\frac{2\pi i}{3}\phi} & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}\phi} & 0 \\ 0 & 0 & e^{-\frac{4\pi i}{3}\phi} \end{pmatrix}. \quad (19)$$

The conjugacy class $C_{m_\phi}^{id}$ is 4-dimensional and it is often referred to as being non-regular because the centralizer $\text{Cent}^{id}(m_\phi)$ is non-Abelian. Indeed, $\text{Cent}^{id}(m_\phi) = SU(2) \times U(1)$ where $SU(2)$ is embedded in $SU(3)$ as in (18) and $U(1)$ is embedded as (19) with ϕ varying. If both string end-points live on $C_{m_\phi}^{id}$ then the gauge group $K_{m_\phi m_\phi}^{id} = SU(2) \times SU(2) \times U(1) \times U(1)$ is 8-dimensional and the naive counting of dimensions gives a hope to bring χ on the slice $\chi^{cc} = \chi$. To see that this indeed happens we use the following decomposition (cf. [14]) of the general $SU(3)$ element:

$$e^\chi = h_L \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{-i\alpha} \end{pmatrix} h_R,$$

where $h_{L,R}$ are of the form (18). Since the matrix in the middle of the decomposition is real and all other matrices are elements of $\text{Cent}^{id}(m_\phi)$ we see that every χ can be brought onto the slice $\chi^{cc} = \chi$ by the gauge transformation (17).

We finish with two comments on quantization: 1) First of all, among the branes leading to the nested duality there are such that satisfy the quantization condition of the single-valuedness of the path integral (cf. [1, 10]). This is certainly true for the case where one end-point of the open string sticks

on the element c of the group center but also for the case when both endpoints are attached to $C_{m_\phi}^{id}$. Indeed, in the latter case, it is sufficient to look at Figure 4 of Ref. [13] where it is described the quantum moduli space of id -branes for several lowest levels k (they coincide with the affine dominant weights and are depicted by small black disks). With varying ϕ , our conjugacy class $C_{m_\phi}^{id}$ sweeps the boundary of the triangular Weyl alcove, hence it intersects the quantum moduli space. 2) Another quantum issue concerns the effective field theory of the open string massless modes. In particular, it would be interesting to relate the results of [15] and of [16] via the nested T-duality.

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